3. The jet in the first part of a swing is due to the local current created by the local difference of pressure; the vortex in the second part of the swing is due to the conjunction of the main stream with the opposing local current set up by the local pressure difference.

My warm thanks are due to Mr. MacKinney for the zeal and ability he displayed in taking the instantaneous photographs from which figs. 3–6 are reproduced.

DESCRIPTION OF PLATES.

Fig. 3.—Obstacle with Pressure and Stream Indicators in Position in Tank, with the Water at rest.
Fig. 4.—Swing from Left to Right—First Part.
Fig. 5.—Swing from Left to Right—Second Part.
Fig. 6.—Swing from Right to Left.

Some Problems Illustrating the Forms of Nebulæ.

By George W. Walker, A.R.C.Sc., M.A., F.R.S., formerly Fellow of Trinity College, Cambridge.

(Received March 27, 1915.)

The possible forms of distribution of a mass of gaseous material under the influence of its own gravitation are of considerable interest in the nebular theory. The law of density which it appears most reasonable to assume is Boyle's Law, in which the pressure is proportional to the density, unless the pressure becomes so great that the material begins to resemble an incompressible substance. Although it is unlikely that the temperature is uniform throughout, still the solution under this restriction would be of value as a step in the direction of greater knowledge as regards possibilities in astronomical phenomena.

The equations can be formed and lead to a differential equation for the surfaces of equal density. This equation is not linear, and in the three-dimensional case little progress to a general solution has been made. In the two-dimensional case, however, considerable progress can be made.

A number of years ago I obtained the exact solution of the statical case of symmetry about the origin. Shortly after I found that Pockels had obtained the complete solution of the statical two-dimensional equation, and

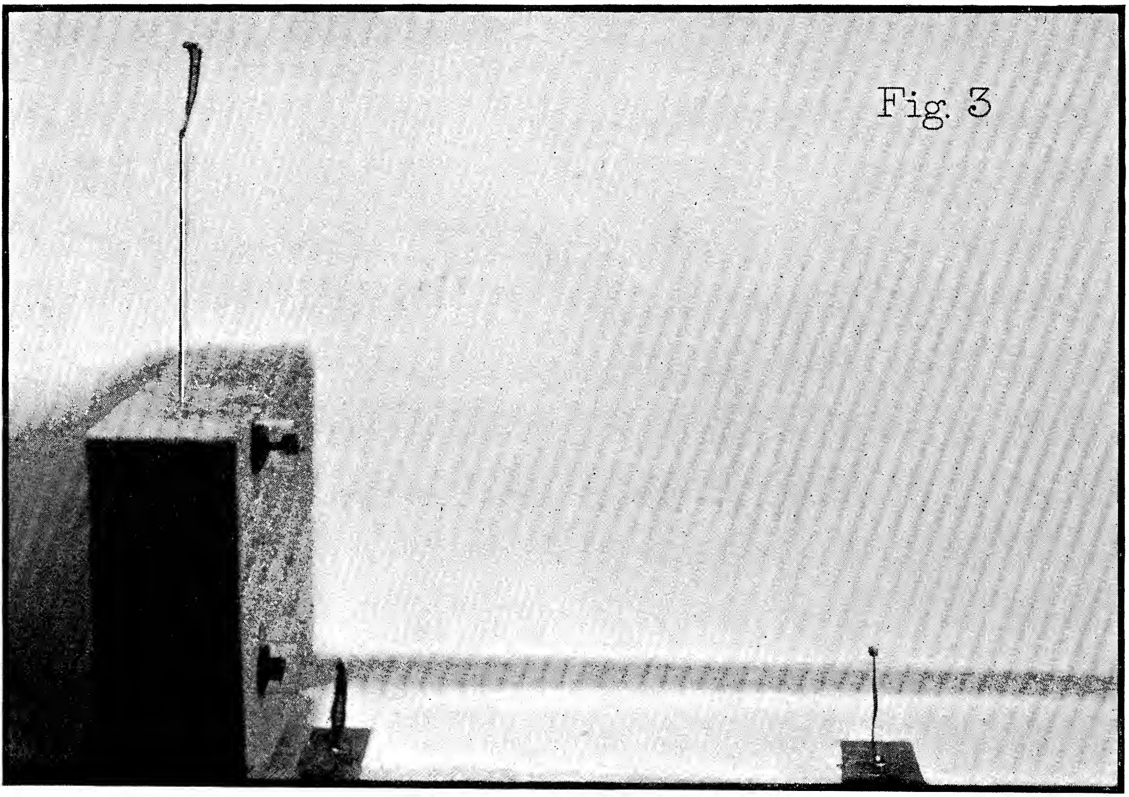


Fig. 3

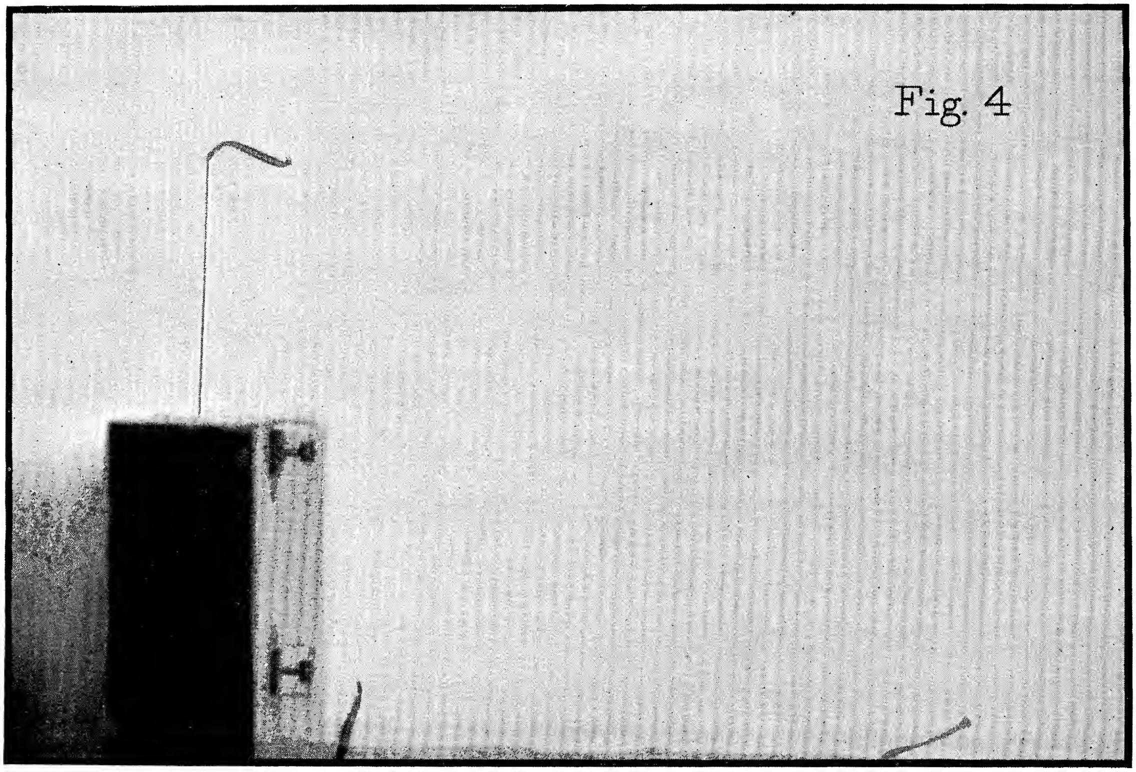


Fig. 4

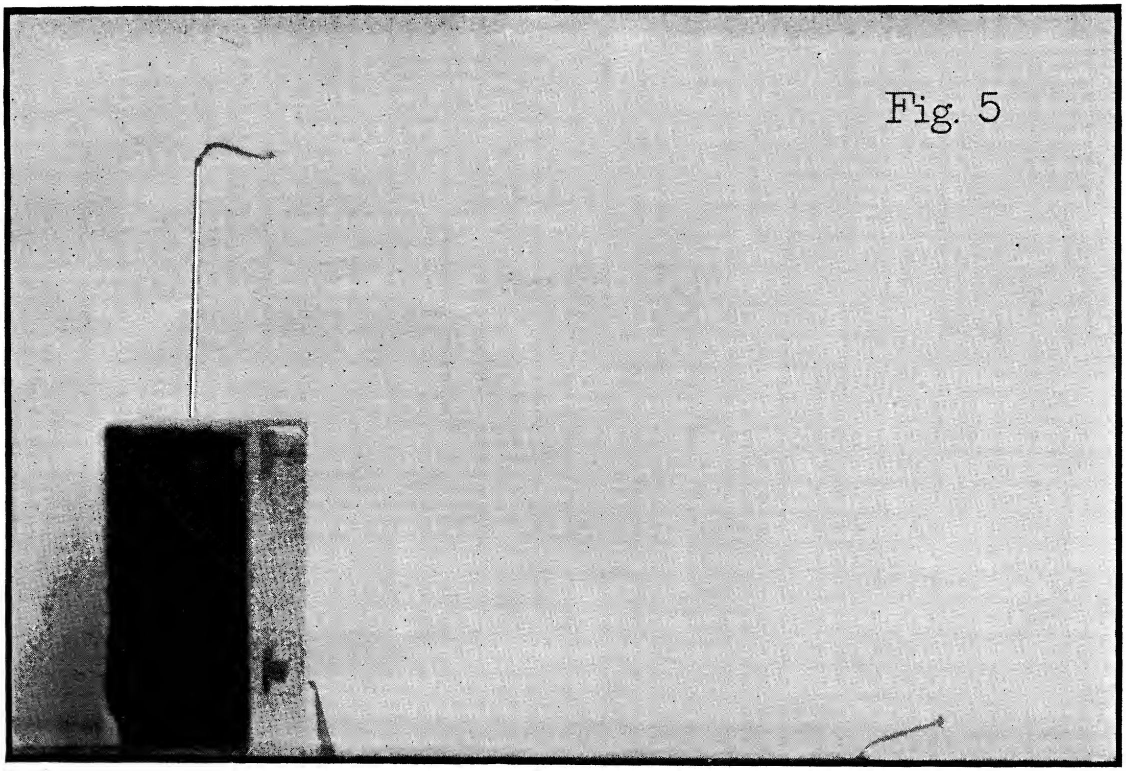


Fig. 5

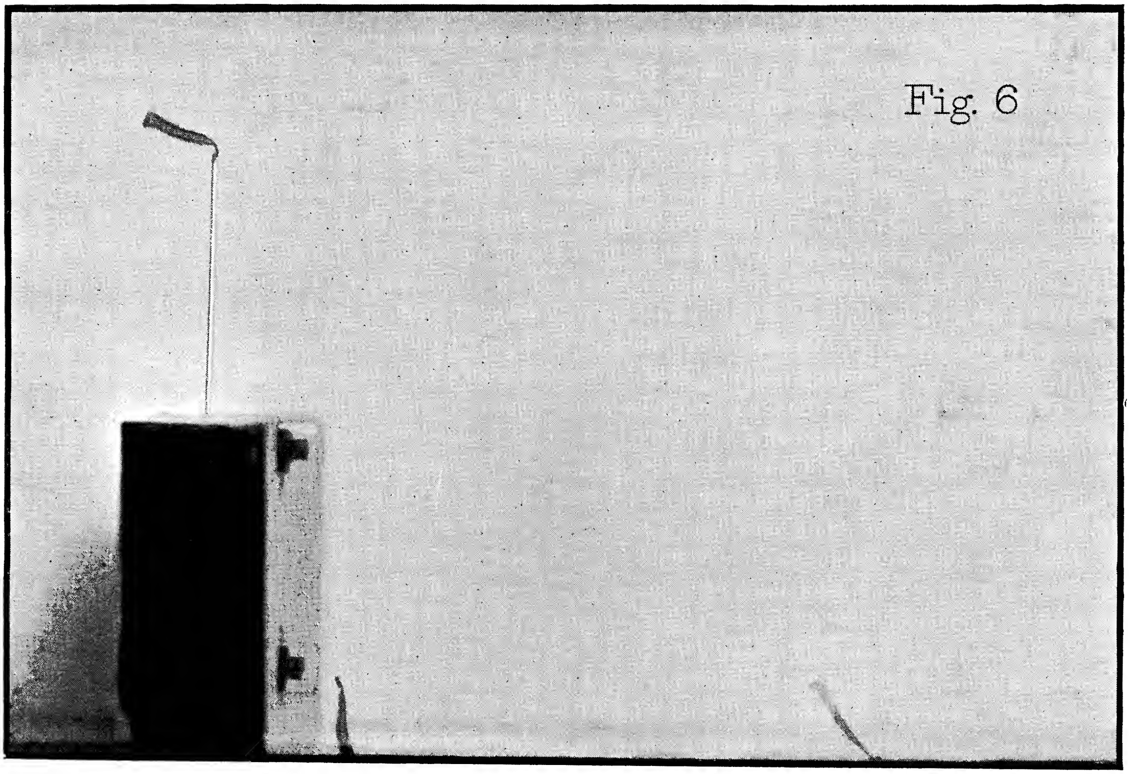


Fig. 6

I drew attention to this in a short paper contributed to the 'Boltzmann Jubilee Volume,' p. 242, but, so far as I know, no further application of the solution obtained by Pockels has been made. I have recently examined some particular cases and have arrived at results indicating such a close analogy with actual astronomical forms that I venture to hope they may be of interest as suggesting types that may be expected in three dimensions.

We begin by consideration of the fundamental equations. Let V be the gravitational potential, γ the gravitational constant, p the pressure at any point of the gas, ρ the density at any point of the gas, x, y, the Cartesian co-ordinates of any point.

The hydrostatic equations of equilibrium are

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right) = \left(\frac{\partial \mathrm{V}}{\partial x}, \frac{\partial \mathrm{V}}{\partial y}\right),$$

and the potential must further satisfy the equation

$$\frac{\partial^2 \mathrm{V}}{\partial x^2} + \frac{\partial^2 \mathrm{V}}{\partial y^2} = -4\pi\gamma\rho.$$

If $p = \rho/h$, where h is a constant, we get as an integral

$$1/h \, . \log \rho = \mathrm{V} + \text{constant},$$

or

$$\rho = \rho_0 e^{h\mathrm{V}}.$$

Hence

$$\frac{\partial^2 \mathrm{V}}{\partial x^2} + \frac{\partial^2 \mathrm{V}}{\partial y^2} = -4\pi\gamma\rho_0 e^{h\mathrm{V}}.$$

This is the equation of which Pockels has obtained the solution in terms of two arbitrary functions as follows. As $\chi = f_1(x+iy) + f_2(x-iy)$ is the solution of $\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = 0$, then the solution for V and ρ is

$$\rho = \rho_0 e^{h\mathrm{V}} = -\frac{1}{2\pi\gamma h \chi^2}\left\{\left(\frac{\partial \chi}{\partial x}\right)^2 + \left(\frac{\partial \chi}{\partial y}\right)^2\right\}.$$

We require to put this in a form which gives real positive values to ρ, and I find that this can be secured by taking

$$\chi = f_1(x+iy) + \frac{1}{f_1(x-iy)}.$$

Otherwise, if ϕ and ψ are conjugate functions, so that

$$\phi + i\psi = f_1(x+iy),$$

we get

$$\chi = \phi + i\psi + \frac{1}{\phi - i\psi},$$

and hence

$$\rho = \rho_0 e^{h\chi} = \frac{2}{\pi\gamma h} \frac{\{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2\}}{(\phi^2 + \psi^2 + 1)^2}.$$

This remarkably general form gives us possible surfaces of equal density, when any known conjugate functions are assigned. I do not know whether a more general form giving real positive density exists, and it may be a matter of interest to pure mathematicians to investigate this point, just as it would be of value to applied mathematicians to know the most general form.

Case I.—If we take

$$\phi = (r/a)^n \cos n\theta, \qquad \psi = (r/a)^n \sin n\theta,$$

we get

$$\rho = \rho_0 e^{hV} = \tfrac{1}{2}\frac{n^2}{\pi\gamma h a^2}\frac{a^2}{r^2}\operatorname{sech}^2 n \log (r/a)$$

$$= \frac{2n^2}{\pi\gamma h a^2}\frac{a^2}{r^2}\left\{\left(\frac{r}{a}\right)^n + \left(\frac{a}{r}\right)^n\right\}^{-2}.$$

This is the most general case of circular symmetry.

We note that when $n < 1$ the density becomes infinite at the origin, elsewhere finite, and vanishes at ∞. When $n = 1$ the density is finite at the origin, and elsewhere is finite, vanishing at ∞. When $n > 1$ the density is zero at the origin, rises to a finite maximum, and then diminishes and vanishes at infinity.

Writing the equation in the form

$$\rho/\rho_1 = 4\frac{a^2}{r^2}\left\{\left(\frac{r}{a}\right)^n + \left(\frac{a}{r}\right)^n\right\}^{-2}.$$

the curves in fig. 1 show ρ/ρ_1 as a function of r/a, for the four special values $n = 0, \tfrac{1}{2}, 1$, and 2; a is still at our disposal while $\rho_1 = 2n^2/\pi\gamma h a^2$ is determined when n and a are fixed.

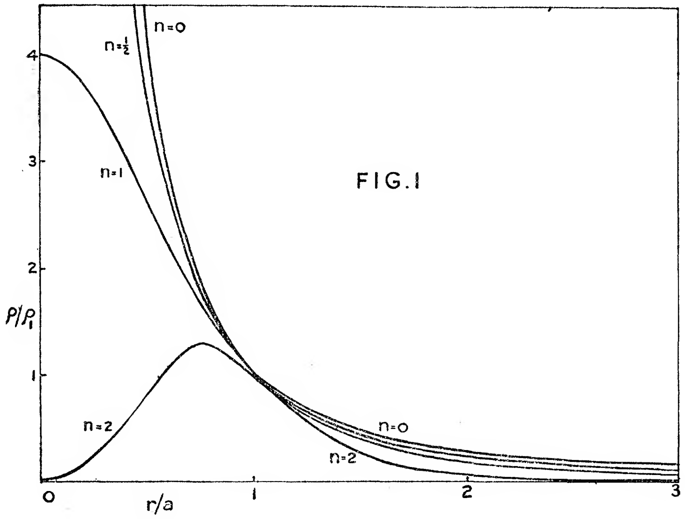


FIG. 1

Boyle's Law involves the possibility of an infinite density, and the law does not hold physically so far. In this and in other cases where a singularity occurs we may replace it by a solid nucleus arranged so as to give the proper value of V over its surface.

This case of circular symmetry for $n > 1$ appears to have some analogy to a ring nebula.

Case II.—Take

$$\phi = b^{-1}\log r/a, \qquad \psi = b^{-1}\theta,$$

we get

$$\rho = \rho_0 e^{h\mathrm{V}} = \frac{2b^2}{\pi\gamma h a^2}\frac{a^2}{r^2}\left[\log^2(r/a)+\theta^2+b^2\right]^{-2}.$$

The surfaces of equal density are given by

$$\theta^2 = -\log^2\frac{r}{a} - b^2 + \left(\frac{2b^2}{\pi\gamma h a^2\rho}\right)^{\frac{1}{2}}\frac{a}{r}.$$

The curve, fig. 2, shows the particular surface

$$\theta^2 = -\log^2\frac{r}{a} - 1 + \frac{a}{r},$$

drawn to a scale of 10.

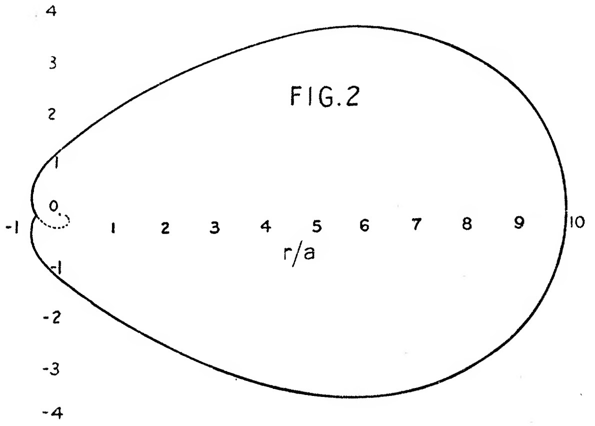


FIG. 2

This case is somewhat peculiar. Starting with $\theta = 0$ when $r/a = 1$, the curve for positive values of θ proceeds as shown, continually approaching the origin by a succession of diminishing spirals. The locus is, however, symmetrical about $\theta = 0$. If now we proceed to draw another surface of different density it would be found to cross the original curve at a succession of points for which $\theta > \pi$. This would mean that at such points two densities are possible, and we cannot admit this. We must, therefore, stop the curve at the nodal point $\theta = \pi$, and then the complete series of curves of equal density form a series of non-intersecting curves of this remarkable pear shape, the density falling off to zero at infinity and increasing towards the origin with a singularity at the origin itself. The pear may be made sharper or blunter by taking different values of b, which is at our disposal.

This form of distribution is of some interest in connection with the researches of Darwin, Poincaré, and Jeans.

Case III.—Let us take the elliptic co-ordinate transformation

$$x+iy = c\cosh(\xi+i\eta),$$

and let

$$\phi+i\psi = e^{-n(\xi-\xi_0+i\eta)}.$$

Then we get

$$\rho = \rho_0 e^{h\mathrm{V}} = \frac{n^2}{2\pi\gamma hc^2}(\sin^2\eta+\sinh^2\xi)^{-1}\operatorname{sech}^2 n(\xi-\xi_0).$$

Thus the surfaces of equal density may be written in the form

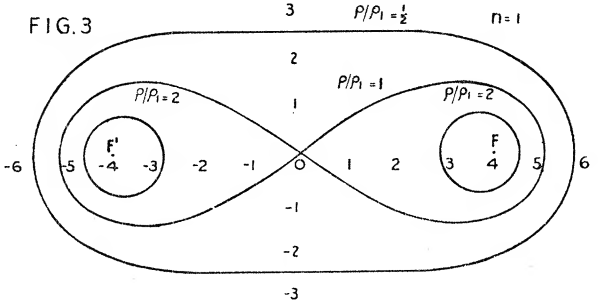
$$\sin^2\eta = \frac{\rho_1}{\rho}\frac{1}{\cosh^2 n(\xi-\xi_0)}+1-\cosh^2\xi.$$

For $n = 1$, $\xi_0 = 0$, we get

$$\sin^2\eta = \frac{\rho_1}{\rho}\frac{1}{\cosh^2\xi}+1-\cosh^2\xi.$$

Fig. 3 shows the curves obtained for $\rho/\rho_1 = \frac{1}{2}$, 1, 2. The density falls off to zero towards infinity, while for values $\rho/\rho_1 > 1$ the locus breaks up into two oval curves about F and F′, which are the foci of the original ellipses in the transformation. F and F′ are singularities at which ρ would be infinite. The general character of the loci is not altered by giving ξ_0 some finite value. For comparison fig. 4 shows on the same scale the result of putting $n = 2$. The equation is

FIG. 3



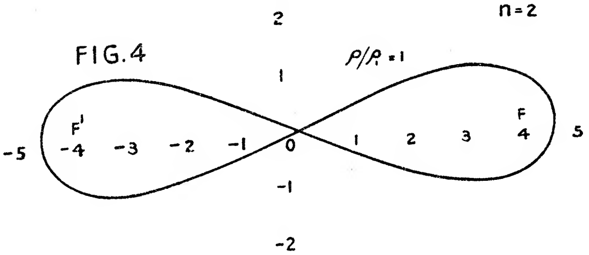
$$\sin^2\eta = \frac{\rho_1}{\rho}\frac{1}{\cosh^2 2\xi}+1-\cosh^2\xi,$$

and the curve drawn is for $\rho/\rho_1 = 1$. It is seen to be rather flatter than the corresponding curve for $n = 1$.

This case then, as a whole, has some analogy in the case of a nebula with two fundamental equal nuclei.

Numerous other cases may be worked out, and it is simply a question of detail in assigning forms to the conjugate functions ϕ and ψ. But perhaps these three cases are sufficient to indicate the generality of the method, and the remarkable interest of the forms that can be obtained.

FIG. 4



I pass to the consideration of an important related problem.

The material of an actual nebula may be moving and not at rest, and in particular it may be rotating. As the rotation, if it exists at all, is very slow, the divergence from the statical equilibrium in such a case would be extremely small, so that the statical solutions are themselves of value. It is, however, of interest to know what the effect of motion may be.

The simplest case we can consider is the final state in which the material rotates about an axis like a rigid body, with steady angular velocity ω. There the problem reduces to a statical one, when we take axes rotating with the material and add to the gravitational forces the centrifugal effect. In the case of a gas such a rotation, however small, can hardly be expected to extend indefinitely, and we should rather expect the motion to fall off and finally cease at a great distance. While keeping this in view, it is not without interest to examine what effect such an imposed uniform rotation would have. Mr. Jeans* has referred to this problem, but without giving details. Some of my results are in agreement with his conclusions and others apparently diverge.

If the components of velocity are u and v, and we neglect viscosity, the equations for a steady motion referred to fixed axes are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{h}\frac{\partial \log \rho}{\partial x}+\frac{\partial \mathrm{V}}{\partial x},$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{1}{h}\frac{\partial \log \rho}{\partial y}+\frac{\partial \mathrm{V}}{\partial y}.$$

$$\frac{\partial}{\partial x}\rho u+\frac{\partial}{\partial y}\rho v = 0.$$

$$\frac{\partial^2 \mathrm{V}}{\partial x^2}+\frac{\partial^2 \mathrm{V}}{\partial y^2} = -4\pi\gamma\rho.$$

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* 'Phil. Trans.,' vol. 213, p. 462 (1914).

In the case of symmetry, where ρ is a function of r only, we may take

$$u = -yf, \qquad v = xf,$$

where f is a function of r only.

The equations then reduce to

$$\rho = \rho_0 e^{h(\mathrm{V}+\int f^2 r dr)},$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \mathrm{V}}{\partial r} = -4\pi\gamma\rho.$$

If $f = \omega$, we get the case of rigid body rotation, and $\rho = \rho_0 e^{h(\mathrm{V}+\frac{1}{2}\omega^2 r^2)}$.

If $f = \omega a^2/r^2$, we get an "irrotational" motion, and $\rho = \rho_0 e^{h(\mathrm{V}-\frac{1}{2}\omega^2 a^4/r^2)}$.

In the latter case the velocity falls off as r increases but is infinite at the origin. We might make f any function of r; and among these we may select one, *e.g.* $\omega e^{-r/a}$, which makes the velocity zero at the origin and also zero at great distances. But such cases are not steady and the influence of viscosity would arise.

We cannot, in general, integrate in finite terms the equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \mathrm{V}}{\partial r} = -4\pi\gamma\rho_0 e^{h(\mathrm{V}+\int f^2 r dr)},$$

but if we suppose f so small that V differs from its value V_0 in the statical case $f = 0$ by a small quantity χ we have approximately

$$r\frac{\partial}{\partial r} r \frac{\partial \chi}{\partial r} = -\frac{2n^2}{\cosh^2 n \log(r/a)}(\chi + \int f^2 r\, dr),$$

while $$\rho = \tfrac{1}{2}\frac{n^2}{\pi\gamma h a^2}\frac{a^2}{r^2}\frac{1}{\cosh^2 n\log(r/a)}\{1+h(\chi+\int f^2 r\, dr)\}.$$

Let $\chi + \int f^2 r\, dr = z$, and $r = ae^\theta$, then the equation may be transformed to

$$\frac{\partial^2 z}{\partial \theta^2} = -\frac{2n^2}{\cosh^2 n\theta} z + a^2 \frac{\partial}{\partial \theta} e^{2\theta} f^2.$$

Now the equation

$$\frac{\partial^2 z}{\partial \theta^2} + \frac{2n^2}{\cosh^2 n\theta} z = 0$$

has a solution $z = \mathrm{A}\tanh n\theta$.

Hence the equation may be written*

$$\coth n\theta \frac{\partial}{\partial \theta}\tanh^2 n\theta \frac{\partial}{\partial \theta} z \coth n\theta = a^2 \frac{\partial}{\partial \theta} e^{2\theta} f^2,$$

and the particular integral which we require is

$$z = a^2 \tanh n\theta \int \coth^2 n\theta \int \tanh n\theta \frac{\partial}{\partial \theta} f^2 e^{2\theta}\, d\theta\, d\theta.$$

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[*] *Cf.* Boole, 'Differential Equations,' p. 205.

For uniform rotation $f = \omega$, and then

$$z = 2a^2\omega^2 \tanh n\theta \int \coth^2 n\theta \int \tanh n\theta e^{2\theta}\, d\theta\, d\theta.$$

In the case $n = 1$ this gives

$$z = \frac{a^2\omega^2}{2}\left\{ 4\log(y+1) + \frac{(y-1)}{(y+1)}\int^y \left[1 - \frac{2}{y}\log(y+1)\right] dy \right\},$$

and for $n = 2$,

$$z = \frac{a^2\omega^2}{2}\left\{ 2\left(\tan^{-1}y - \frac{y}{1+y^2}\right) + \frac{(y^2-1)}{(y^2+1)}\int^y \left(1 - \frac{2}{y}\tan^{-1}y\right) dy \right\},$$

where $y = r^2/a^2$.

These curves are shown to scale in figs. 5 and 6, and we note that in both cases $z = 0$ when $r = 0$, is everywhere positive, and finally becomes infinite and proportional to r^2/a^2 when r is great.

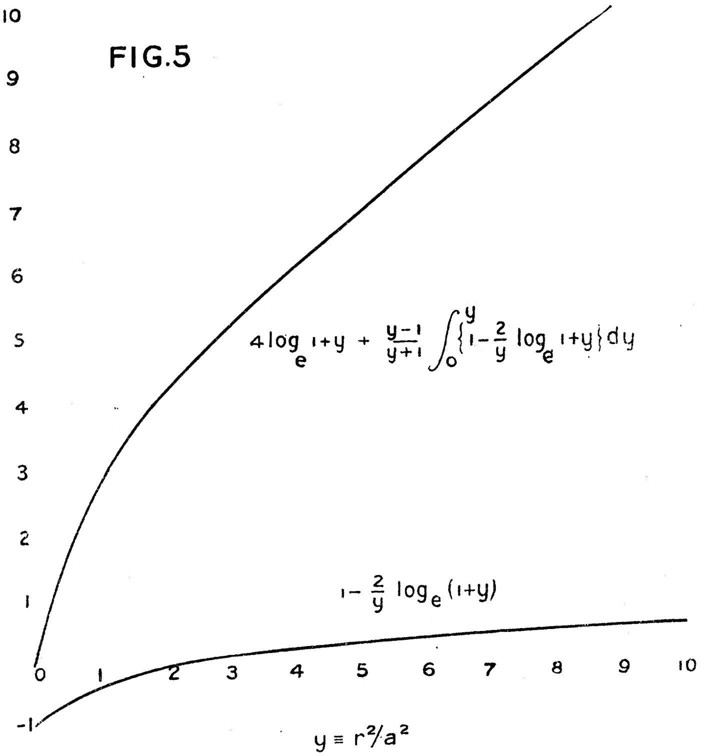
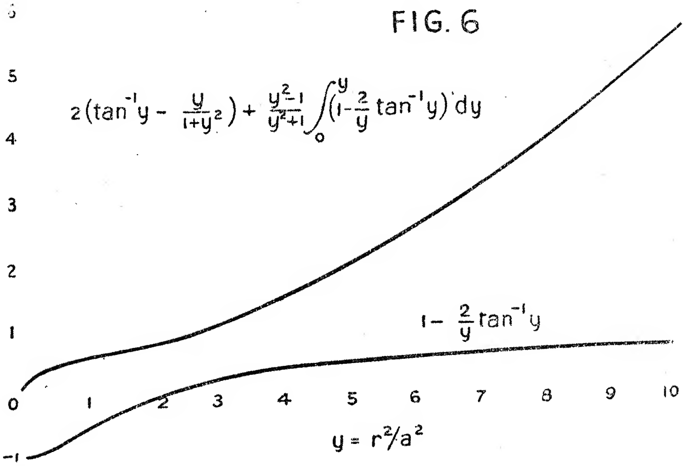


FIG. 5

The general effect of the rotation is then to increase the density from what would obtain in the statical case, and by increasing proportion as r increases from the origin, but even if it were permissible to pass to large values of r the density would still diminish to zero. We cannot, however, extend our solution to large values of z, but we may say that the tendency of the rotation is (*e.g.* when $n = 2$) to make the density rise more steeply to its maximum

FIG. 6



value and to move the point of maximum density further out with increasing ω; but there is no indication of a tendency to throw off the outer layers. This latter result disagrees with the general conclusion stated by Jeans (*loc. cit. ante*).

For "irrotational" motion $f = \omega a^2/r^2$ and the particular integral we require is

$$z = \omega^2 a^2 \left\{ \frac{1}{2y}\frac{(3y^2-1)}{(y^2+1)} + \tan^{-1}y - \frac{(y^2-1)}{(y^2+1)}\int^y \frac{\tan^{-1}y}{y}\,dy \right\},$$

where $y = r^2/a^2$.

In this case z is negative and infinite when $r = 0$, rises to a maximum positive value, and again declines to a negative and infinite value as r becomes indefinitely great. As before, we cannot extend our solution to great values of z, but we may say that the tendency of the motion is to make the distribution hollow near the origin, to increase the concentration towards the maximum density at a finite value of r/a and finally to tend to throw off the outer layers when r becomes great. This latter conclusion is in agreement with Jeans.

When we pass to other cases such as Cases II and III, where the original distribution of density is not symmetrical about the origin, it is rather difficult to think of any steady motion other than uniform rotation which would leave the distribution in anything like a permanent form. We must, therefore, confine our attention at present to this case, artificial though it may be.

Taking axes through the origin and rotating with the material with angular velocity ω, we have the equations

$$\rho = \rho_0 e^{h(\mathrm{V}+\frac{1}{2}\omega^2 r^2)},$$

$$\frac{\partial^2 \mathrm{V}}{\partial x^2}+\frac{\partial^2 \mathrm{V}}{\partial y^2} = -4\pi\gamma\rho.$$

Again, let ω be small and let V differ from its value V_0 when $\omega = 0$ by a small quantity χ. Then

$$\rho = \rho_0 e^{h\mathrm{V}_0}\{1+h(\chi+\tfrac{1}{2}\omega^2 r^2)\},$$

and

$$\frac{\partial^2 \chi}{\partial x^2}+\frac{\partial^2 \chi}{\partial y^2} = -4\pi\gamma\rho_0 h\, e^{h\mathrm{V}_0}(\chi+\tfrac{1}{2}\omega^2 r^2),$$

or, if $\chi+\frac{1}{2}\omega^2 r^2 = z$,

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = -4\pi\gamma\rho_0 h\, e^{h\mathrm{V}_0} z + 2\omega^2.$$

In Case II of the distribution of pear-shaped character I find that the solution will have to be obtained by a somewhat complicated series, and as I have not yet gone very far with it I pass to Case III, which can be integrated in finite terms. We had as the statical solution

$$\rho = \rho_0 e^{h\mathrm{V}_0} = \frac{n^2}{2\pi\gamma h c^2}(\sin^2\eta+\sinh^2\xi)^{-1}\operatorname{sech}^2 n\xi.$$

Hence the equation we have to solve is

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = -\frac{2n^2}{c^2}z(\sin^2\eta+\sinh^2\xi)^{-1}\operatorname{sech}^2 n\xi + 2\omega^2,$$

or on changing to elliptic co-ordinates ξ, η, we get

$$\frac{\partial^2 z}{\partial \xi^2}+\frac{\partial^2 z}{\partial \eta^2} = -\frac{2n^2}{\cosh^2 n\xi}z+\omega^2 c^2(\cosh 2\xi-\cos 2\eta).$$

The solution is $z = z_1+z_2\cos 2\eta$, where z_1 and z_2 are functions of ξ only, and satisfy

$$\frac{\partial^2 z_1}{\partial \xi^2}+\frac{2n^2 z_1}{\cosh^2 n\xi} = \omega^2 c^2\cosh 2\xi$$

and

$$\frac{\partial^2 z_2}{\partial \xi^2}-\frac{2n^2(2\cosh^2 n\xi-1)}{\cosh^2 n\xi}z_2 = -\omega^2 c^2 \qquad \text{respectively.}$$

A solution for z_1, when $\omega = 0$, is $z_1 = \mathrm{A}\tanh n\xi$. Hence the particular integral is

$$z_1 = \omega^2 c^2\tanh n\xi\int\coth^2 n\xi\int\tanh n\xi\cosh 2\xi\, d\xi\, d\xi.$$

A solution for z_2, when $\omega = 0$, is $z_2 = \mathrm{A}\cosh^2 n\xi$. Hence the particular integral is

$$z_2 = \frac{\omega^2 c^2}{n^2}\{\tfrac{1}{6}+\tfrac{1}{3}\cosh^2 n\xi\log\cosh n\xi-\tfrac{1}{6}n\xi\tanh n\xi(2\cosh^2 n\xi+1)\}.$$

The case $n = 2$ admits of easy expression, and we have

$$z_1 = \frac{\omega^2 c^2}{4}(\cosh 2\xi - 2 \operatorname{sech} 2\xi)$$

$$z_2 = \frac{\omega^2 c^2}{4}\{\tfrac{1}{6} + \tfrac{1}{3}\cosh^2 2\xi \log \cosh 2\xi - \tfrac{1}{3}\xi \tanh 2\xi (2\cosh^2 2\xi + 1)\}.$$

We may trace the effect of this on the density. Along the axis of x from O to F (see fig. 4) $\xi = 0$, and η changes from $\frac{1}{2}\pi$ to 0.

Hence $$z = z_1 + z_2 \cos 2\eta = -\tfrac{1}{4}\omega^2 c^2 (1 - \tfrac{1}{6}\cos 2\eta),$$
and so z ranges from $-\frac{7}{24}\omega^2 c^2$ at O to $-\frac{5}{24}\omega^2 c^2$ at F.

From F to ∞, $\eta = 0$ and ξ ranges from 0 to ∞. Hence

$$z = z_1 + z_2 = \tfrac{1}{4}\omega^2 c^2 \{\cosh 2\xi - 2 \operatorname{sech} 2\xi + \tfrac{1}{6} + \tfrac{1}{3}\cosh^2 2\xi \log \cosh 2\xi - \tfrac{1}{3}\xi \tanh 2\xi (2\cosh^2 2\xi + 1)\}$$

This function is $-\frac{5}{24}\omega^2 c^2$ at F, increases and becomes positive as we increase ξ, and becomes infinite and positive with $\omega^2 c^2/4 \cosh 2\xi$ as ξ becomes infinite. Again, along the axis of y, $\eta = \frac{1}{2}\pi$, so that

$$z = z_1 - z_2 = \tfrac{1}{4}\omega^2 c^2 \{\cosh 2\xi - 2 \operatorname{sech} 2\xi - \tfrac{1}{6} - \tfrac{1}{3}\cosh^2 2\xi \log \cosh 2\xi + \tfrac{1}{3}\xi \tanh 2\xi (2\cosh^2 2\xi + 1)\}.$$

At the origin $\xi = 0$ this function is $-\frac{7}{24}\omega^2 c^2$, it increases as ξ increases and becomes positive, finally being infinite and positive with $\frac{1}{4}\omega^2 c^2 \cosh 2\xi$ as ξ becomes infinite.

The effect of the rotation is then to diminish the density near the origin and to increase it at greater distances, but there is no clear tendency to throw off the outer layers, since the function becomes positive and infinite and not negative and infinite. This is, again, in apparent conflict with Jeans' conclusion. It would thus appear from the cases considered here that a finite boundary of a gaseous mass is not to be looked for as a consequence of rigid body rotation, but rather as a consequence of some other type of motion in which viscosity may play a part.

My sincere thanks are due to Sir Joseph Larmor for kindly criticism and advice.

[*Note added April,* 1915.—It has been pointed out to me that in Case II $\partial V/\partial\theta$ becomes discontinuous along the line $\theta = \pm\pi$. Hence this is a line of singularities. In accordance with the principle already stated we must, therefore, suppose this line to be occupied by solid matter of the proper density at any point, determined by the change of $1/r \,.\, \partial V/\partial\theta$ in crossing this line. Theoretically this distribution would extend indefinitely, but practically the density of solid matter required at any point on $\theta = \pi$ becomes very small as the distance from the origin increases.]

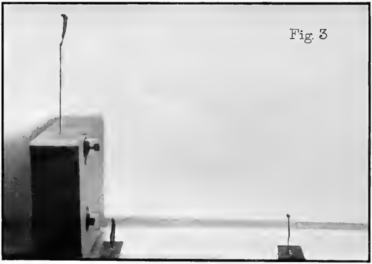


Fig. 3

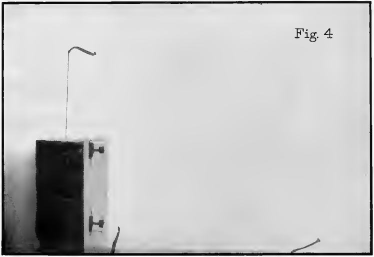


Fig. 4

Fig. 3.—Obstacle with Pressure and Stream Indicators in Position in Tank, with the Water at rest.

Fig. 4.—Swing from Left to Right—First Part.

Fig. 5

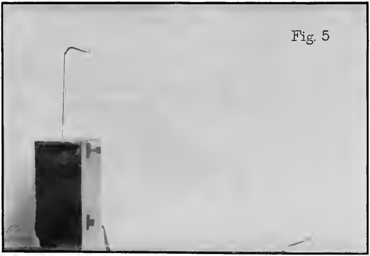


Fig. 6

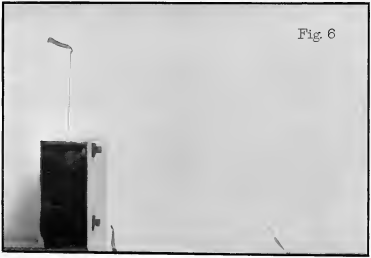


Fig. 5.—Swing from Left to Right—Second Part.
Fig. 6.—Swing from Right to Left.